1. A group of n 2 people decide to play an exciting game of Rock-Paper Scissors. As you may recall, Rock smashes Scissors, Scissors cuts Paper, and Paper covers Rock (despite Bart Simpson saying “Good old rock, nothing beats that!”). Usually, this game is played with 2 players, but it can be extended to more players as follows. If exactly 2 of the 3 choices appear when everyone reveals their choice, say a, b 2 {Rock, P aper, Scissors} where a beats b, the game is decisive: the players who chose a win, and the players who chose b lose. Otherwise, the game is indecisive and the players play again. For example, with 5 players, if one player picks Rock, two pick Scissors, and two pick Paper, the round is indecisive and they play again. But if 3 pick Rock and 2 pick Scissors, then the Rock players win and the Scissors players lose the game. 1 Assume that the n players independently and randomly choose between Rock, Scissors, and Paper, with equal probabilities. Let X, Y, Z be the number of players who pick Rock, Scissors, Paper, respectively in one game.   
   (a) Find the joint PMF of X, Y, Z.   
   (b) Find the probability that the game is decisive. Simplify your answer (it should not involve a sum of many terms).   
   (c) What is the probability that the game is decisive for n = 5? What is the limiting probability that a game is decisive as n ! 1? Explain briefly why your answer makes sense.

Let's tackle the problem step-by-step:

**(a) Joint PMF of XXX, YYY, ZZZ**

**Definitions:**

* XXX: Number of players who pick Rock.
* YYY: Number of players who pick Scissors.
* ZZZ: Number of players who pick Paper.

Since each of the nnn players independently chooses Rock, Scissors, or Paper with equal probability 13\frac{1}{3}31​, XXX, YYY, and ZZZ follow a multinomial distribution:

The joint PMF of XXX, YYY, and ZZZ is given by:

P(X=x,Y=y,Z=z)=n!x! y! z!(13)x(13)y(13)zP(X = x, Y = y, Z = z) = \frac{n!}{x! \, y! \, z!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^zP(X=x,Y=y,Z=z)=x!y!z!n!​(31​)x(31​)y(31​)z

where x+y+z=nx + y + z = nx+y+z=n and x,y,z≥0x, y, z \geq 0x,y,z≥0.

**(b) Probability that the Game is Decisive**

The game is decisive if exactly 2 of the 3 choices appear. This can happen in two ways:

1. X>0X > 0X>0, Y>0Y > 0Y>0, Z=0Z = 0Z=0 (Rock and Scissors only).
2. X>0X > 0X>0, Y=0Y = 0Y=0, Z>0Z > 0Z>0 (Rock and Paper only).
3. X=0X = 0X=0, Y>0Y > 0Y>0, Z>0Z > 0Z>0 (Scissors and Paper only).

**Probability Calculation:**

1. **Probability of Rock and Scissors only (Z = 0):**

P(Z=0)=∑x+y=nn!x! y! 0!(13)x(13)y(13)0=(13)n∑x+y=nn!x! y!=(13)n⋅2nP(Z = 0) = \sum\_{x + y = n} \frac{n!}{x! \, y! \, 0!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^0 = \left(\frac{1}{3}\right)^n \sum\_{x + y = n} \frac{n!}{x! \, y!} = \left(\frac{1}{3}\right)^n \cdot 2^nP(Z=0)=x+y=n∑​x!y!0!n!​(31​)x(31​)y(31​)0=(31​)nx+y=n∑​x!y!n!​=(31​)n⋅2n

1. **Similarly, the probability for Rock and Paper only (Y = 0):**

P(Y=0)=(13)n⋅2nP(Y = 0) = \left(\frac{1}{3}\right)^n \cdot 2^nP(Y=0)=(31​)n⋅2n

1. **Similarly, the probability for Scissors and Paper only (X = 0):**

P(X=0)=(13)n⋅2nP(X = 0) = \left(\frac{1}{3}\right)^n \cdot 2^nP(X=0)=(31​)n⋅2n

**Total Probability:** Since these events are mutually exclusive:

P(Decisive)=3⋅(13)n⋅2n=3⋅2n3n=3⋅2n3nP(\text{Decisive}) = 3 \cdot \left(\frac{1}{3}\right)^n \cdot 2^n = \frac{3 \cdot 2^n}{3^n} = \frac{3 \cdot 2^n}{3^n}P(Decisive)=3⋅(31​)n⋅2n=3n3⋅2n​=3n3⋅2n​

**(c) Probability for n=5n = 5n=5 and Limiting Probability**

**For n=5n = 5n=5:**

P(Decisive for n=5)=3⋅2535=3⋅32243=96243≈0.395P(\text{Decisive for } n = 5) = \frac{3 \cdot 2^5}{3^5} = \frac{3 \cdot 32}{243} = \frac{96}{243} \approx 0.395P(Decisive for n=5)=353⋅25​=2433⋅32​=24396​≈0.395

**Limiting Probability as n→∞n \to \inftyn→∞:** As nnn grows, the probability of exactly 2 of the 3 choices appearing (which makes the game decisive) tends to 1. This is because the number of players choosing any given option becomes very large relative to the number of options, so it's almost certain that some choices will be made, and some will not.

Thus, the limiting probability that the game is decisive as n→∞n \to \inftyn→∞ approaches **1**.